UNIVERSITY OF CAMBRIDGE
Faculty of Education

ISSN 2043-8338

Journal of Trainee Teacher Education Research

A critical analysis of how the use of variation theory affects Year 8 mixed-ability students' development of proportional reasoning

Ruth Ní Mhuircheartaigh

(PGCE Secondary Mathematics, 2023-2024)

email: m98ruth@gmail.com

https://doi.org/10.17863/CAM.128705

Abstract

In recent years, variation theory has gained popularity as a mathematics pedagogy which develops conceptual and procedural fluency. Despite a lack of research in the English classroom context and in specific areas of the mathematics curriculum, variation theory is one of the Five Big Ideas underpinning the Teaching for Mastery programme. This action research addresses these gaps by investigating the use of variation theory in teaching proportion in a Year 8 classroom. Adopting an action research approach, lessons were planned and delivered using variation theory; data was collected before and after by interviewing four pupils, and from teacher observation. This research concludes that variation theory supports the formation of a relational and instrumental understanding of proportion, with particularly positive effects for lower-attaining students.

A critical analysis of how the use of variation theory affects Year 8 mixed-ability students' development of proportional reasoning

Ruth Ní Mhuircheartaigh

Introduction

In recent years, variation theory has gained international popularity as an approach to teaching and learning. Emerging from the phenomenographic tradition, variation theory was originally introduced as a theory of perceptual learning (Marton et al., 1997). The simple guiding principle, that a concept can only be discerned when variation in the concept is experienced, is captured by the Chinese maxim "only by comparing can one distinguish" (p.14) which has underpinned the use of a variation theoretic pedagogical approach in several East-Asian countries for several decades (Gu et al., 2017). Given these countries' successes in international assessments, e.g. Programme for International Student Assessment (PISA), other countries including England have become interested in the implementation of such approaches.

Researchers have expressed concern about attempts to 'borrow policy' without due consideration for the complex nature of classroom learning (Jerrim & Vignoles, 2016). However, as the situation stands, the National Council for Excellence in Teaching of Mathematics Mastery programme includes variation theory as one of its Five Big Ideas. It is therefore imperative that the applicability of variation theory to English classrooms be investigated.

Currently, variation theory is posited as a general approach to teaching; however, its applicability to specific learners and to specific content is not addressed (Kullberg et al., 2017). These are major shortcomings of the research currently existing in this area. The goal of this study is to address these shortcomings in a small-scale but detailed enquiry. The aims of this action research are two-fold: to investigate the viability of variation theory as a method of teaching ratio and to ascertain to what extent variation theory can support proportional reasoning skills grounded in conceptual understanding in a Year 8 mixed-ability lesson. Given the unusual opportunity to investigate a pedagogical approach in a mixed-ability setting, I will also examine my findings through this lens.

Firstly, I will review literature on variation theory. Following this, the methodological design of this action research will be discussed in depth. Drawing on the data collected, I will then present and discuss the findings of this research.

Review of the literature

In the spirit of variation, I begin by outlining how variation theory differs from other theories of mathematics education. Since the peak of behaviourism's popularity has subsided, theories have generally been either individualist or collectivist (or an attempt to marry the two). The former is characterised by Piaget's work suggesting that a concept is constructed individually by the learner, and the latter by Vygotsky who argued that meaning is co-constructed by individuals in an environment influenced by individuals' skills, speech, and cultures. While powerful ways of viewing a classroom in their own right, these theories may be considered weakened by a lack of account of what is being taught (Runesson, 2005). Variation theorists, on the other hand, focus their attention on the *object of learning* (what is being taught).

To provide a comprehensive overview of variation theory, the literature studied in the following section features a number of different approaches and types of intellectual project: a learning study undertaken in Hong Kong featuring student data (Marton & Pang, 2006), a learning study undertaken in Sweden (Kullberg et al., 2017), a theoretical knowledge-for-critical-evaluation study about task design (Watson & Mason, 2006) and a theoretical knowledge-for-understanding study on the role of procedural variation in Chinese education (Lai & Murray, 2012). I will then discuss the claims made by Runesson et al. (2005) about the compatibility of variation theory with the interactionist theories of teaching and learning.

Conceptual variation and exposition

Variation theorists posit that to experience a critical aspect of a topic, one must experience variation in that aspect; in pedagogical terms, learners' attention must be drawn to the critical aspect being varied. Marton & Pang's (2006) learning study in Hong Kong tracks the learning of five different class groups with five different teachers and concludes that a precise regime of variation and invariance is necessary for pupils to access and master a specific object of learning. Their research is structured by three different phases of how content is delivered in a classroom: the *intended, enacted* and *lived* objects of learning. These are elucidated by the following questions: "what is intended to be learnt?" (meaning facts and skills); "what is made possible to learn?" and "what is learnt?" (Runesson, 2005, p.70).

The lived object of learning was studied by comparing the responses of 77 students from the target group and 96 from the comparison group to two problems at the end of the lesson sequence; pupils in the target group had been taught by teachers who had experience using variation theory in their teaching and were aware of the theory underlying it, while the comparison group were taught by teachers not explicitly aware of the link between variation and learning. A statistically significant relationship was established between more sophisticated responses to the problems and being in the target group. To understand the differences, the intended and enacted objects of learning were studied by lesson plan examination and the lesson observations respectively. While all five teachers' lesson plans featured variation theory to some extent, e.g. the effects of change in demand and change in supply on price of various goods were considered separately by both target and comparison group teachers, there were several differences. For example, the target group used the same good in all examples and when showing the effect of variation of a change in supply/demand, the effects of increase and decrease were exemplified whereas the comparison group only encountered the effects of increase in supply/demand. While the lesson plans were similar, the enacted object of learning differed in terms of how efficiently the target group teachers drew attention to the patterns of variation present.

As discussed by Kullberg et al. (2017), a variation theory teacher begins by deciding what the critical aspects of a topic are which leads them to open certain dimensions of variation. In their learning study investigating the changes in a teacher's lessons on the concept of equality before and after the teacher gained an in-depth awareness of variation theory, they found that each of the lessons (before and after) featured conceptual variation but with different critical aspects being made discernible. However, the conceptual variation in the lesson following the intervention was far better-suited to the aims of the teacher; following the identification of critical aspects, he was newly able to draw pupils' attention to the critical aspects he had identified during the planning stage. There are then more decisions to be made which may make it less or more possible for pupils to experience a pattern of variation in an active way; in Marton and Pang's (2016) study, these decisions made by teachers, e.g.

concerning the use of technology and group-work, were subordinate to the pattern of variation and invariance. The continued importance of such decisions is highlighted by Marton and Pang's (2016) finding that one target group was more successful than the other, despite their lesson plans being almost identical.

If a pupil successfully discerns a particular pattern of variation, a teacher may be able to predict the response of a pupil - this is the basis of Watson and Mason's (2005) argument that the use of variation can allow teachers to predict hypothetical learning trajectories. This conclusion rests on the assumption that critical aspects are equally accessible and likely to be discerned by all pupils. While Marton and Pang (2006)'s findings compellingly suggest the usefulness of planning lessons around the critical aspects to be discerned, Watson and Mason's (2005) suggestion seems less sensible when one considers, as is the case with my research, mixed-attainment class groups. This concern is reflected by Kullberg et al. (2017) in their suggestion that "what critical aspects are for different groups of learners" (p.567) be explored by future research.

Procedural variation and mathematical behaviour

Repetition in exercises is a point of contention in mathematics education. Beyond consensus around the obvious point that the more often you encounter a concept or procedure, the easier it is to recall, attitudes toward repetition are divided. Albeit with exceptions, this divergence in scholarly opinion appears to have a West-East dimension (Lai & Murray, 2012). In Western mathematics education literature, repetitive practice is often construed as a remnant of the drill-and-practice tradition of behaviourist teaching and therefore ill-suited to modern constructivist classroom learning (Lehtinen et al., 2017). In Chinese mathematics education, in particular, repetition is seen to support deep understanding; one mechanism by which deep understanding is facilitated via repetition is in the use of procedural variation in exercises (Lai & Murray, 2012).

Procedural variation is the systematic varying of a problem with the aim of inducing a step-by-step formation of knowledge by drawing learners' awareness to parts of the relevant concept. Gu et al. (2004) describe three main ways in which this can be done: varying the conditions of a problem while keeping the problem and method the same; varying the method while keeping the problem invariant; varying the problem itself while using the same method. Lai & Murray (2012) give the following example (Figure 1, next page) of the first of these types:

Problem 1: There are 9L of apple juice and every 3L is put in a jar. How many jars are needed?

Problem 2: There are 9L of apple juice and every 1L is put in a jar. How many jars are needed?

Problem 3: There are 9L of apple juice and every 0.3L is put in a jar. How many jars are needed?

Problem 4: There are 9L of apple juice and every 0.1L is put in a jar. How many jars are needed?

Problem 5: There are 9L of apple juice and every 0.05L is put in a jar. How many jars are needed?

Figure 1: measurement problem with procedural variation.
[Redrawn from “Teaching with Procedural Variation: A Chinese Way of Promoting Deep Understanding” by Lai & Murray (2010, p.10)]

Firstly, note how all elements of the problem remain invariant (including the wording of the problem) except the amount per jar (the divisor) here. Secondly, the systematic decrease of the divisor and effect on the answer may lead the learner to conjecture and then to attempt generalisation, e.g. the smaller the divisor, the bigger the quotient (Lai & Murray, 2012). This exercise elucidates how a well-designed variation exercise can support the formation of both instrumental and relational understanding (Skemp, 1978); the repetition allows for the former while the controlled pattern of variation affords learners the opportunity the latter. The second type of procedural variation described by Lai & Murray (2012) is the variation of method, exemplified by solving a pair of simultaneous equations. By using three methods (elimination, substitution, and graphical methods) contemporaneously, a connected understanding of the algebraic and graphical methods can be achieved.

Considering these two examples, one may argue that learners can approach these with the sole aim of completing the procedure to get the right answer. Indeed, the progressions of a pupils’ relational understanding and conjecturing cannot be assumed; there is consensus across the literature that the role of the teacher is crucial (Lai & Murray, 2012; Watson & Mason, 2006). Lai and Murray (2012) highlight how effective scaffolding in conjecture-building will support learners in conjecturing. In turn, Watson and Mason (2006) highlight the teacher’s role in helping learners view variation exercises, like those in Figure 1, as a mathematical object – that is, viewing exercises together rather than as distinct and disconnected. By prompting pupils to do so and to ask questions such as “what changes and what stays the same?" (Watson & Mason, 2006, p.18), they are afforded the opportunity to experience the pattern and result of the variation.

Compatibility of variation theory with other theories

As discussed above, beyond the exposition of a particular topic, a teacher must consider how classroom organisation align with the aims of a lesson. One central consideration is that of interaction – this may be between pupils themselves, between pupils and teacher or in Borba & Villarreal's (2005) more expansive realm of humans-with-media. Variation theory, in its original form as a theory of perceptual learning and as we have encountered it so far, appears to fit within the constructivist paradigm: the teacher opens dimensions of variations and if pupils discern them, they are afforded the opportunity to make sense of patterns and related critical aspects (Lai & Murray, 2012).

This view of variation theory is challenged in Runesson's (2005) reappraisal of Jaworski's (1999) ethnological study of six mathematics teachers, each of whose teaching is informed by constructivism (either radical, social, or Piagetian). Runesson (2005) conducts a variation-theoretic study of the transcriptions from Jaworski's (1999) lesson observations and concludes that variation theory can complement interactionist or constructivist theories. In one dialogic episode between pupils and teacher where the teacher's questions about similar shapes constitute scaffolding within the learners' zone of proximal development, Runesson (2005) reveals how the questions are of a variation theoretic form – the teacher probes the pupils on what stays the same and what changes between two similar shapes. In this way, the questions the teacher can open dimensions of variation. The research of Runesson (2005) and Kullberg et al. (2018) highlight the importance of which questions are asked: differences in how the object of learning is enacted implies differences in the lived object of learning.

In other excerpts from (Jaworski, 1999)'s study, Runesson (2005) shows how questions prompting pupils to consider variation and invariance serve as a springboard for conversation between pupils and an opportunity to construct meaning of a mathematical concept together. The possibility of integrating variation theory into a social constructivist classroom is also considered by Barton (n.d.)'s online resources and Watson and Mason (2005) who suggest the question "what changes and what stays the same?" as, respectively, a prompt for discussion and as a metacognitive mantra.

Summary and research questions

There are three central themes in the literature reviewed. The first two of these are the theory and practice of, respectively, conceptual and procedural variation. The final theme is the perspective of the object of learning as co-constituted by interactionist and variation theoretic accounts. Marton & Pang (2006) show that if a teacher draws learners' attention to a critical aspect of a concept by using controlled variation, this critical aspect can be discerned by a striking number of pupils. While Marton & Pang (2006) note that conceptual variation may be visible in the lessons of teachers who are ignorant of variation theory, Kullberg et al. (2017) show that a teacher's ability to plan and view topics through a variation theoretic lens and to use conceptual variation efficiently is positively affected by increased awareness of variation theory.

Lai & Murray (2012) and Watson & Mason (2006) exemplify how exercises featuring procedural variation support pupils' development of deep and relational understanding in a given topic. Moreover, Lai & Murray (2012) challenge the pervasive stereotype of Chinese education as uniquely focused on drill and memorisation by highlighting the extent to which repetition coupled with procedural variation can foster deep understanding in pupils.

Finally, Runesson (2015) highlights how the use of variation theory can be paid due attention while also attending to the classroom environment and how meaning is constructed; Runesson (2015)'s demonstration of variation theory's amenability to a social constructivist classroom is particularly compelling.

My first Research Question (RQ) was devised in order to address the gap in the literature about variation theory's applicability to specific areas of the national curriculum:

> RQ1: Can variation theory support the learning of ratio?

Given the importance of proportion in England's national curriculum coupled with its status as a watershed concept and capstone of mathematics learning (Fernandez et al., 2010; Lesh et al., 1988), pupils' development of relational understanding of proportion is highly desirable. Therefore, I will also seek to answer the question:

> RQ2: In what ways does variation theory support the development of a relational understanding of proportion?

Methodology

This research was conducted in a small mixed gender school of around 700 pupils in Cambridgeshire. The percentages of EAL students and students with SEN support both exceed the national average while the number of pupils eligible for free school meals is below the national average. Most remarkable is the school's commitment to mixed ability teaching with setting implemented only in line with GCSE examinations; the Year 8 class with whom this research was undertaken is a mixed ability class of 28 pupils.

The methodology of action research was chosen for this study due to its following the four steps of plan-act-observe-reflect typical of action research. While action research is often concerned with ameliorating a certain aspect of current practice, this action research is primarily concerned with viability (of the use of variation theory) rather than improvement of current practices. Qualitative data was collected via interviews and lesson observations. I decided to focus exclusively on qualitative data due to variation theory's emergence within the phenomenographic tradition; phenomenography is concerned with the qualitatively various ways in which people experience an event or concept. I will now outline the research design, beginning with ethical considerations, in advance of describing the data collection and analysis in further detail.

Ethical Considerations

A major part of this research was the consideration of an ethic of respect. I consulted British Educational Research Association's (BERA) most recent guidelines (BERA, 2018) as well as relevant literature on educational research (Cohen et al., 2018). This involved developing an awareness of the potential issues for confidentiality and integrity wrought by my dual role (BERA, 2018); moreover, all my interactions with pupils were informed by this. I also consulted the school's ethics literature and sought permission to record interviews.

To obtain informed consent, participants in interviews were informed about what would be involved in interviews, that there would be no negative implications of refusing to participate and that they were free to stop participating at any time without explanation (BERA, 2018). To ensure anonymity and to give participants some ownership over their contributions, I invited them to choose their own pseudonyms. To guard against possible obstacles to confidentiality which may arise from participants choosing names of those close to them (Allen & Wiles, 2016), I asked pupils to choose pseudonyms not personally familiar to them.

Creating a comfortable and non-harmful environment for research subjects, both passive and active, is an essential component of ethical research. While planning interviews and teaching, I was highly cognisant of the way in which the affective domain of learning permeates mathematics learning. This was demonstrated in the interview design; additionally, lessons were planned in such a way not to import harm from variation theory's lack of account of different types of learner.

Lesson design

An understanding of ratio is central to the development of proportional reasoning skills; ratio is introduced in primary school and re-visited each year to inculcate a more deep and sophisticated understanding of proportion. The expectations of a Key Stage 3 pupil, as laid out by the national curriculum, are outlined below in the third column of Table 1.

Lesson	Lesson plan: activities and content	Corresponding Key Stage 3 national curriculum content
1	Retrieval starter: mixed content Exposition and worked examples (I do, You do) featuring conceptual variation in the representation of ratio (ratio notation, visual, fractional) Independent practice featuring procedural variation in finding equivalent ratios Problem solving in turn-and-talk format with focus on communication and reasoning	Ratio, proportion, and rates of change – use ratio notation; relate the language of ratios and the associated calculations to the arithmetic of fractions
2	Retrieval starter: mixed content Exposition featuring conceptual variation in equivalent ratios Procedural variation exercises on MWBs: equivalent fractions (AfL) Worked examples (I do, You do): finding parts of equivalent ratios Independent practise featuring procedural variation in finding parts of equivalent ratios Worked examples (I do, You do): representing multiplicative relationships using ratios Problem solving, featuring procedural variation, in turn-and-talk format with focus on reasoning and communication: deciphering between additive and multiplicative problems (AfL)	Ratio, proportion, and rates of change – use ratio notation, including reduction to simplest form Understand that a multiplicative relationship between two quantities can be expressed as a ratio Solve problems involving percentage change, including percentage increase, decrease and original value problems.

Table 1: Outline of the lessons with use of variation theory and relevant national curriculum objectives identified

The planning of lessons for this action research was informed by Watson and Mason’s (2005) suggested series of processes for planning variation theoretic lessons. Their suggested steps are to analyse the topic; identify critical aspects in that topic; identify variations that would help pupils discern these and decide how and which dimensions of variation to open; construct exercises featuring controlled, procedural variation which may lead pupils to discern the associated critical aspects; provide sequences of activities based on series of hypothetical responses to variation. Given my prior experience teaching the Year 8 class coupled with the comments of Kullberg et al. (2018) concerning variation theory and the associated literature’s lack of consideration of the myriad types of mathematics learners present in any (mixed-attainment) classroom, I decided to omit the final step of this process. An outline of the two lessons can be found in the table below; the lessons will be discussed in further detail in the Findings and Discussion section.

Credibility and trust-worthiness

In recent decades, interpretivist researchers have directed significant attention to the question of how to evaluate qualitative research conducted in naturalistic settings. The ‘scientific holy trinity’ of validity, reliability and generalisability has been rejected in the domain of interpretivist research due to, among other reasons, naturalistic settings being highly changeable which renders reliability an ill-fitting goal (Kvale, 1995). Among the alternative evaluation criteria is Lincoln and Guba (1985)’s trustworthiness which hinges on credibility and transferability.

Credibility concerns truthfulness and the correspondence between a researcher’s communication of participants’ viewpoints and the viewpoints themselves, it is therefore also concerned with inner validity of an experiment’s methodology. Actions taken to enhance credibility include the triangulation of my observation and deliberate selection of interviewees, the observing teacher’s observations and the pupils’ reflections; this has the added benefit of producing a comprehensive perspective on the lessons (Patton, 1999). These are discussed in further detail in the next section.

Transferability refers to the applicability of findings to other settings. To produce high-quality research, and to fulfil one’s responsibility to other researchers, a researcher must clearly detail the conditions under which the research was conducted. I have attempted to do this in all aspects of the description of this research.

Overview of data collection methods

Within the variation framework, discernment implies an experience of variation. If we assume that people can only express what they have learned or discerned, what is expressed corresponds to what has been opened as a dimension of variation. As variation theory is a theory of discernment, it is essential to hear directly from the learners. Young people, particularly those in schooling, are more familiar with verbal self-expression than written; additionally, there are aspects of self-expression such as analogy which are helpful to understanding someone's experience which are more likely to be used verbally than in writing (Collier-Reed et al., 2009). For these reasons, collecting data via interview was an obvious choice. In addition to interviews, observations were used to gain an external perspective of how the object of learning was enacted in this specific classroom.

Interviews

The interviews were conducted with four pupils and each lasted twenty minutes. An assumption of one piece of guiding literature is that learners have a natural propensity to spot and understand patterns (Watson & Mason, 2006). It was beyond the scope of this research to consider whether this was true in the relevant classroom setting but given the complex relationship between mathematics learning and affective aspects such as motivation, I was not comfortable with this assumption. Therefore, in choosing participants, I deliberately chose pupils who I was confident possessed this propensity and enacted it in lessons. Coupled with the aim of illustrating the effects of variation theory in a mixed-ability classroom, I picked pupils who were representative of a range of attainment levels. The first interview took place the day before the first lesson and the final interview took place on the Monday following the final lesson on the Friday. This sequence of interviews and lessons was designed to ensure that the pupils' reflections in the final interview would only relate to the relevant lessons and that the time between the final lesson and interview was minimised.

The main motivation for using group interviews was to make the interview setting less intimidating for the participants. Another benefit of the group interview is that the interaction between group can lead to a more authentic response being co-constructed and given by the group than would be the case with an answer given directly to the interviewer. One group interview was carried out the day before the first lesson and the second interview was carried out on a Monday following the second lesson on the previous Friday.

“To make the strange familiar,” (Morrison, 2013, p.321) I shared the purpose of the interviews with them and we spoke about educational research. In hindsight, I believe this had the added benefit of the participants understanding that they played an important role in the interviews and participating in a co-operative manner, reflecting Jansen’s (2015) finding that children may be eager to participate in interviews as they feel that their views and experiences are valued. Two disadvantages of group interviews are the tendency for group-think to occur and for one or two participants to dominate conversation; conceivably, the likelihood of these is exacerbated when the discussion centres on mathematics as pupils may experience feelings of intimidation or have an impression that they need to provide a “correct” answer. To avoid these issues arising, I cued people into the conversation in different orders when it was appropriate and put considerable thought into the way I reacted to pupil responses in advance of the interviews.

Given an awareness of the main topics I wanted to discuss with the pupils as well as an awareness of my prior lack of insight into the pupil’s experiences, the interview was modelled on Patton’s (1980) interview guide approach where questions were asked in an open manner. In addition to promoting conversation, this reduced the response bias which may arise from closed questions. In the spirit of curiosity and recognition of the invaluable contribution of the pupils’ reflections, I maintained an openness to new data and to the emergence of new phenomena (Kvale, 1996). For example, when discussing the pupils’ experiences of doing questions, I had not planned to discuss challenge; however, following one pupil invoking of challenge and others responding to this pupil, it was fruitful and interesting to discuss others’ views on how they experience challenge.

When discussing mathematics, I was careful to use language which I was confident was familiar to the participants. In addition to the pre-planned and spontaneous questions arising from topics broached by pupils, two types of prompts were used: confirmatory probes were used to ensure credibility in interviews and expansive probes were used to gain further information (Cohen et al., 2018). Some of these were planned and others were spontaneous. The specific aims and findings of the interviews will be discussed in the Findings and Discussion section. An overview of the key topics, planned questions and probes for each interview can be found in Table 2 (next page).

Pre-lessons interview	When do you like mathematics best? (What types of activities? What level of challenge?) How do you feel when a topic makes sense to you? (More confident doing exercises? More enjoyable?) What helps you make sense of a topic? What is your experience of doing exercises (independent practice) in lessons? What do you think it proportion and ratio are all about? Does it make sense? Please take two minutes to consider this question [basic proportional reasoning problem] and how you would find an answer. You can write down your workings if it would help you to explain to us [After two minutes of looking at two problems – one additive, one multiplicative] Tell me what you think (Are there differences between these questions? Would you approach these in the same way?)
Post-lessons interview	[Reflecting on lesson one with lesson materials on ratio] Tell me about doing these exercises (Did you spot any patterns in these exercises? OR Did you learn anything new while doing these exercises?) [Reflecting on lesson two with lesson materials on equivalent ratios] Did you learn anything from this whiteboard exercise? (Do you notice anything about the whiteboard exercises?) [After two minutes considering three questions - two multiplicative, one additive] Tell me what you think (Can you say something in general about what types of question ratios can be used to solve?) What is ratio and proportion all about? (Has anything changed in your view of these topics?)

Table 2: Overview of pre- and post-intervention interview questions (and probes)

Observation

Variation theory is concerned with how the object of learning is handled. As highlighted by Kullberg et al. (2017) and Marton & Pang (2006), different teachers, or the same teacher in different lessons, will make different aspects of the same topic discernible to learners. Therefore, having an observer who can attest to how the object of learning is enacted in the classroom of interest is an essential perspective. Utilising observation data also introduces a 'reality check' into the reflection stage of action research (Robson, 2002 as cited in Cohen et al., 2018), increasing the credibility of the study.

In both lessons, the class teacher observing made extensive notes of what they had observed. The presence and data provided by this teacher were particularly valuable due their prior familiarity with the class and ability to notice changes in behaviour (mathematical or otherwise) and evaluate engagement. In general, my attention was fully on the class; however, I made quick short-hand notes of three of my observations during the lessons. To minimise data loss and to ensure my observations were independently made, I immediately wrote down my reflections before discussing the lessons with the observing teacher.

Data analysis

I arranged the interviews so that I would have time immediately after to reflect and to begin transcribing from the recordings made. This was helpful as I was able to recall tone of voice and body language of pupils and annotate the transcription with these. While the scope of this research did not necessitate transcription, it was helpful to do so as having a legible representation of the interview makes it easier to identify themes (Cohen et al., 2018). Following transcribing and identifying themes in both interviews, a process of iterative coding was used. While themes were often assigned to a 2-3 minute section of interview, codes were attached to single sentences; this increase in granularity risks potential bias in interpretation and loss of the essence of the interview (Cohen et al., 2018). To mitigate against this, I returned to the original transcriptions and recordings frequently. When this process was completed, the transcriptions were deleted in line with the school's policy.

In the first lesson, there were two observers present; in the second lesson, there was one. Each observer, on both occasions, gave me verbal and written feedback shortly after the lesson. Following each lesson, after I had written my own reflection, I identified similarities and differences in our observations. The final step of analysing the observation data was to revisit it following interviews, e.g. observations which sought to describe the behaviour during specific points in the lesson were cross-checked with how the pupils' experiences. In keeping with the view of Cohen et al. (2018), this specific method of handling reflections, observations and interview became cumbersome at times; in light of this, I only attempted to do this for certain parts of the lessons.

Findings and discussion

In this section, the main findings are presented, and I will attempt to answer the guiding questions of this research:

RQ1: Can variation theory support the learning of ratio?

RQ2: In what ways does variation theory support the development of a relational understanding of proportion?

Before advancing to the findings, it will be helpful to outline what relational understanding looks like in the context of proportion. During the 1970s and 1980s, there were several influential studies carried out about the learning of ratio. The seminal work of Lesh et al. (1987) focuses on representations of ratio and their role in developing a connected understanding of ratio. The use of multiple representations is widely accepted to facilitate relational understanding across many areas of mathematics, including ratio and proportion. In this study, an understanding of how different representations of ratio are related to one another is considered a relational understanding of proportion as it indicates the development of a schema of knowledge (Skemp, 1978) – here, of pupils' prior learning on fractions, decimals and percentages (FDP) and ratio.

A result of relational understanding is the ability to decipher the applicability of a concept to a given problem (Skemp, 1978). In proportion, a full relational understanding constitutes an ability to distinguish between multiplicative and linear problems and to realise that ratio and proportion are helpful with the former and not the latter (Tourniaire & Pulos, 1985).

In keeping with the variation theoretic framework of this paper, the findings and discussion will be guided by the handling of the object of learning – ratio - with a focus on the critical aspects. Following a description of the object of learning in each lesson composed of reflections, observation data and pupils' reflection, the findings will be investigated for evidence for RQ1 and RQ2. Before discussing the lessons, I will discuss the baseline data of pupils' conceptions of understanding mathematics and of proportion.

Understanding students' understanding

In the pre-lesson interview, I was surprised by the pupils' perceptions of instrumental and relational understanding. Initially, when asked about what they like about mathematics, there was a range of responses. One pupil, Katniss, mentioned that they find mathematics most enjoyable outside of the lesson when they are working on a problem with friends. In contrast with this, Luna said that they prefer it "when we're doing a bunch of questions that we just need to answer in our own time" and expressed a preference for doing questions alone. Robert and Achilles both made comments about completing exercises; while Robert likes "when there's a story behind the question," Achilles likes to know what's going on. This gave rise to a disagreement about whether it is good or not to know *fully* what is going on, naturally invoking the concept of an understanding which goes beyond instrumental.

In response to the question “do you like it when a topic makes sense to you?” the pupils were divided. Katniss nodded in enthusiastic agreement with Luna’s statement “I don’t mind if I don’t know how it works – I just need to know how to do it.” Luna, a few minutes later, offered the following reason:

> *“When we learn the method of doing it in a way “that makes sense” (air quotes the speaker’s), that’s – most of the time – a harder method than if you just have the method of doing it with no reasoning.”*

Robert and Achilles, in a manner of spirited yet friendly debate, expressed their disagreement with Luna and Katniss’ views. Robert spoke of how simply understanding what a question means gives them “a small sense of achievement – like, I’ve actually learned something, supporting Skemp's (1978) assertion that the development of a schema in a certain topic is “intrinsically satisfying” (p.14). Achilles echoed this view, explaining that if you understand what is happening, one “can just get on with it”, and offered the eloquent summary that “otherwise, the task isn’t the questions – it’s to figure out what the question means.”

Students’ initial conceptions of proportion

To gain some sense of pupils’ perception of proportion and a base-line datum of their sense-making, I asked the pupils how they would explain proportion to a friend. Following some confusion by and feigned horror at the idea of engaging in such a conversation with a friend, a co-constituted meaning promptly emerged around the utility of proportion. Achilles was the first to respond with “the best buys and stuff? I think it’s good for life skills.” Katniss and Luna agreed, respectively adding that “it’s relevant to working out prices and stuff” and that “[proportional reasoning] is probably the most important mathematical skill to have […] because you use it in life.” It is unclear to what extent these are the true impressions of the pupils and to what extent these responses are bound up with the myth of utility of mathematics (Andrews, 1998). Perhaps, the myth has been peddled very successfully and the participants do indeed view the topic through a utilitarian lens; another potential consequence of the myth which is relevant in the interview setting is that the pupils may consider it the ‘correct’ response to mention a real-life use of proportion. Either way, this view held by three of the interviewees points to a predominantly instrumental understanding of proportion (as it derives from the procedure of finding prices of n goods given the price of m goods).

In response to their peers, seemingly unenthused by the utility of proportion, Robert asked "is it not just increasing and decreasing things using multipliers… like using fractions and decimals and things?" In the literature on proportional reasoning, Robert's views and allusion to multiplication corresponds to a more sophisticated understanding of proportion (Tourniaire & Pulos, 1985). Moreover, the reference to FDP (fractions, decimals, and percentages) suggests that Robert had already developed a relational understanding of proportion. Indeed, this was confirmed by the pupils' responses to the basic proportional reasoning question:

If three potatoes cost $45p$*, how much will eight potatoes cost?*

On the continuum of stages of proportional reasoning development which ranges from qualitative thinking to build-up strategies to multiplicative reasoning (Tourniaire & Pulos, 1985), the participants' responses all featured build-up strategies and multiplicative reasoning. One pupil used a hybrid strategy of using multiplicative reasoning to find the cost of two potatoes to be 30p, and then adding these up: $30p + 30p + 30p + 30p$. Two pupils used a two-step multiplicative strategy – divide by three, multiply by eight – and one pupil, Robert, used the most efficient one-step strategy of multiplying by $\frac{8}{3}$.

To understand the degree to which the four interviewees had developed this sense of proportional reasoning, they were asked to consider the following questions (Table 3) and how they might approach them:

Ronnie, the miniature dachshund, weighed 2 kg as a puppy. Now Ronnie weighs 5 kg. Ralph, the French bulldog, weighed 9 kg as a puppy. Now Ralph weighs 13 kg. Which dog has grown more?	Two cyclists, Ava and Isaac, are cycling equally fast around the cycling track. Ava began cycling before Isaac arrived at the track and had completed 9 laps when Isaac had completed 3. When Ava has completed 15 laps, how many laps will Isaac have completed?

Table 3: two problems used in the pre-intervention interview to understand pupils' conception of linear versus multiplicative problems

Three of four participants correctly identified that the second question is a linear problem. Only one pupil, Robert, identified multiplicative reasoning as a method to solve the first problem. In conclusion, prior to the intervention, each participant had developed proportional reasoning to a level consistent with the use of build-up or multiplicative reasoning while each pupil's understanding equipped them, at least, to decipher what a linear solution is or, put differently, what a multiplicative solution is not.

By drawing on the triangulated observations of myself and the observing teacher, in addition to the reflections of the four interview participants, I will now discuss the intervention lessons and associated findings.

Effectiveness of variation theory to understanding ratio

At the planning stage, I identified the following critical aspects: what a ratio is, invariance of a given ratio under multiplication and the applicability of ratio to multiplicative problems. A clearer outline of the first of these is needed – *what a ratio is* refers to how ratio representation of part-part and part-whole relationships relates to proportions such as fractions, decimals, and percentages.

To make *what ratio is* discernible to learners, the dimension of variation opened was that of representation; pictorial, formal (a: b), fractional-formal ($\frac{a}{a+b}:\frac{b}{a+b}$), simplified fractional-formal and spoken language were each used. In each worked example and in each exercise (Figure 2), the ratio to be represented was kept invariant.

Visual representation	$x:y$ or $x:y:z$	Ratio of fractions $\frac{}{}:\frac{}{}$	Ratio of simplified fractions
	4:8	$\frac{4}{12}:\frac{8}{12}$	$\frac{1}{3}:\frac{2}{3}$
	5:10		
	3:1	$\frac{3}{4}:\frac{1}{4}$	
x × × × × × × y × ×			

Figure 2: The first four exercises of a Lesson 1 worksheet on ratio which features procedural variation

Prior to the introduction of equivalent ratios in the second lesson, this set of exercises afforded pupils the opportunity to discern another critical aspect - the invariance of ratios under multiplication (scaling). This was done by keeping the simplified fractional-formal representation invariant while the given representations were multiples of one another, e.g. $5:10$ and $4:8$. The range of variation was restricted to small, whole-number multiples to ensure that computation did not become the focus. In accordance with previous comments about the mixed-attainment class group, the following activity did not rely on this being discerned. This affordance (to discern the scaling property) was a secondary function of the exercises and subordinate to the primary purpose of discerning *what ratio is*.

Considering the primary purpose of this first lesson, to make the critical of aspect of *what ratio is* discernible, I will now highlight the key classroom observations before discussing the reflections of the interviewees. Whilst circulating in the room, it was clear that all pupils were able to complete the exercises when the visual or $x: y$ representation of the ratio was given; the repetition combined with a progressive increase in difficulty, with the only given representation being formal-fractional towards the end, provided pupils with an opportunity to develop procedural fluency while experiencing challenge. Additionally, each observer reported a high level of engagement and a positive working environment during the exercises.

At the end of the lesson, pupils worked together on the following two short reasoning problems (Table 4):

James and Cameron are watching a film. They pause it with one seventh left to watch. What is the ratio of how much they have watched to how much is left to watch? a) 1:7 b) 7:1 c) 1:6 d) 6:1	Two pupils, Gemma and Aodh, are arguing about the following information: *On a school trip, the ratio of teachers to students is 1:9* Gemma says that there are nine times as many teachers as students. Aodh says that $\frac{1}{10}$ of the people on the trip are teachers. Who is correct? a) Gemma b) Aodh c) Both Gemma and Aodh d) Neither Gemma nor Aodh

Table 4: two problems used in the second lesson to assess pupil's learning

The purpose of this activity was to assess whether pupils had developed a relational understanding – in this instance, an ability to transition between representations and form an argument based on these. Additionally, the emphasis on reasoning and pair-discussion allowed the pupils to develop their ability to communicate about ratio, another form of representation. Nearly all pupils in the room answered the questions successfully, demonstrating confidence and ease with interpreting the questions and representing fractions as ratios and vice versa; moreover, the highly correct and consistent use of the spoken language representation of ratio, e.g. "parts" and "whole", was notable. These findings are consistent with the findings of Lai and Murray (2012) that the use of various methods for a given problem contributes to a relational understanding; this also supports the finding of Watson and Mason (2006) that repetition in exercises can exercise both relational and instrumental understanding. Notably, there were clear benefits for less mathematically confident pupils. In the two lessons, the observer and I agreed that there was a higher level of engagement than usual in these learners; prior to the first lesson, I had not seen these learners engage so confidently and positively with applying their knowledge to previously unseen questions. The most plausible explanation for this is that the variation theoretic exposition of material and exercise design, by definition, follows the instructional design principles suggested to avoid extraneous cognitive load and extraneous demands on working memory (Sweller, 2016). I tentatively suggest that this aspect of variation theory allowed many pupils in the classroom to engage more deeply with the content than they usually would.

In the post-intervention interview, we reflected on this set of exercises. Three of the four pupils said that they found these exercises helpful for recalling "how ratios work." In contrast with the pre-intervention interview, where the pupils had said that they usually do not experience moments of learning or realisation whilst completing exercises, Luna reported "understanding how ratios and fractions and percentages are connected" and voiced agreement with Achilles who described realising that "all the fractions add to one, so it's just like proportion … each bit of the ratio – or number of parts – is a fraction of the whole number of parts." The pupils' discussion of the ratio lesson as well as the previous topic of FDP strongly suggested that they had developed relational understanding.

While it is possible that the variation theoretic design of the exercises, coupled with the exposition which featured conceptual variation in the representation of ratio, was responsible for the positive effect on relational understanding, there are many other factors at play. It is possible that the pupils had strong prior knowledge of ratio and that the lesson was effective in helping pupils recall this. Alternatively, viewing the situation through an exclusively social constructivist lens, one may argue that the pupils had only developed an instrumental understanding and upon interaction with their peers were able to co-construct a relational understanding.

What can be stated with certainty is that all pupils discerned the first critical aspect. In my previous experience of this class, both as observer and teacher, I had not experienced such a large majority of the pupils actively opting to attempt a question where there was not a clear solution; similarly, this was remarked upon by the observing class teacher. This suggests to me that the pupils had not only discerned the first critical aspect but also developed a degree of relational understanding which equipped them with self-efficacy.

As for the second critical aspect made partially accessible to pupils in this lesson, I asked a random selection of pupils about whether they had spotted any patterns in the exercises whilst circulating. The pupils who had identified and could explain the pattern were the pupils who had completed the worksheet most quickly and among those whom I have previously observed to have the most secure mathematical knowledge. This recalls the concern of Kullberg et al. (2017) that literature on variation theory does not address the fact that some aspects will not be discernible to all learners. It may also be indicative of the comments of Watson & Mason (2006) that pupils generally approach mathematical exercises as discrete and disconnected exercises.

In the second lesson, following the introduction of equivalent ratios, the pupils completed a mini white-board activity (MWB) which involved pupils responding yes/no to whether a given pair of ratios were equivalent or not. There were eight pairs of ratios and a pattern of procedural variation, e.g. the first pair was 2:3 and 4:6 and the next pair was 2:3 and 6:9. Following the suggestion of Watson and Mason (2006), pupils were prompted to consider the exercises as a collection by considering what changes and what stays the same between questions. In this way, if a pupil found that two pairs ($x\colon y$ and $a\colon b$; $x\colon y$ and $c\colon d$) were equivalent, it was possible for pupils to discern that $a\colon b$ and $c\colon d$ are also equivalent. By the fourth pair, I noticed that eight of twenty-eight pupils, including two of the interviewees, were producing correct answers very quickly. When I asked a selection of these pupils to explain to the class how they got their answer, each explained by invoking multiplication and referring to an earlier question. Following the whiteboard activity, pupils completed exercises where they were to find the antecedent/consequent of ratio given the consequent/antecedent and a second ratio to which it is equivalent; to afford pupils another opportunity to discern the critical aspect of the invariance of ratios under multiplication, the ratio with no unknowns was kept invariant throughout. My observations and that of the observing teachers agreed that pupils were successfully completing these exercises, albeit some more easily than others. While this demonstrates that the pupils had achieved an instrumental understanding of invariance of a given ratio under multiplication, it remains unknown to what extent they had developed a more fundamental understanding of the connection between multiplication and ratios.

During the preceding interview we discussed these exercises, Luna mentioned that it was “too easy” because it was “the same as what we did on the whiteboards.” This was a particularly interesting comment because although the exercises to be completed were not the same, the underlying principle of the invariance of multiplication was indeed the same. When I probed their understanding, by asking what they meant by this, they explained that you just had to “multiply or look for multiples” in each. Following this, I asked the other pupils if they had any comments to add about the whiteboard exercises or following exercises and showed them the resources in question to remind them. Luna offered an equally interesting reflection on the whiteboard questions, pointing specifically to the procedural variation present: “I think it’s to train your mind to fixate on one thing – you could figure out if two are equivalent and then use that for the next question.” By offering such an accurate explanation of the procedural variation and the ability to “transfer” knowledge, it was clear that the way Luna had accessed these questions as a structured set had “exposed [them] to mathematical structure affording them enhanced possibilities for making their own sense of a collection of questions.” (Watson & Mason, 2006, p.92)

The second part of the second lesson focused on the third and final critical aspect - the applicability of ratio to multiplicative problems. The dimension of variation opened was types of problem solvable – this is Lai & Murray (2012) *multiple application of method* form of procedural variation. The problems, including similar shapes and price reduction/increase, were all previously familiar to the Year 8 class; within the dimension of variation of these similar problems, the range of variation was linear-multiplicative. I first modelled how ratios are used to represent problems involving two quantities; following this, the dimension of variation was opened via the questions in the slide (presented as Figure 3, next page) which pupils were prompted to discuss.

It was clear to the observer and I that a large majority of pupils were able to identify which questions could be solved using ratios. Pupils were most confident identifying the first question, with the most common explanation offered being “it is like finding multipliers” referring to recent learning about percentage increases/decreases. In brief conversations with a selection of pupils, which intentionally included pupils who had been observed to have a low level of comfort with the topic of proportion over the preceding weeks, all pupils correctly identified that this ratio table does not give the correct solution for the third question.

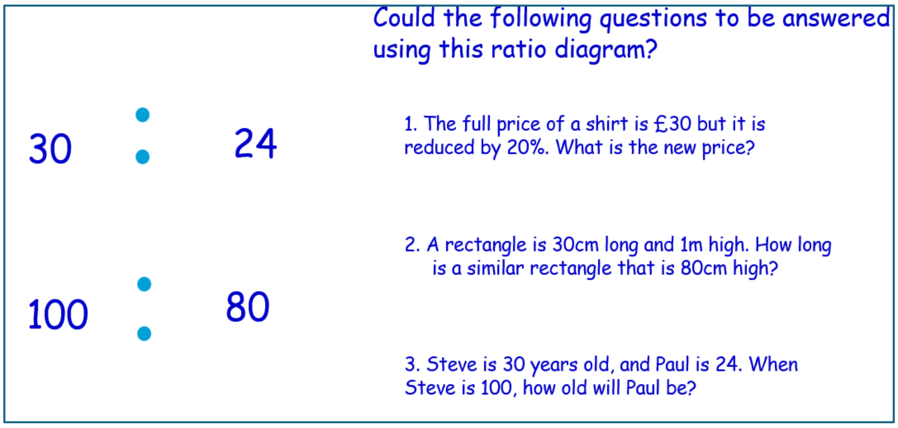

Figure 3: Material from Lesson 2 on linear/multiplicative problems,
[adapted from NCETM Proportional Reasoning resources]

However, none of these pupils were able to identify why this question differs from the others. For these pupils, I conclude that the critical aspect had only been partially discerned. They had ascertained which questions were solvable by this method; however, they had not discerned *why* this was the case – indicating a partially formed relational understanding.

The final section of the second interview was focused on the theme of linear and multiplicative problems. Provided with three questions like those above (see Figure 3), but without the ratio table, all pupils identified which questions could be solved using ratios and accurately described them as linear/multiplicative. Despite there only having been two lessons, it was fascinating and encouraging to witness how all the pupils were now in agreement that proportion is "all about multiplying." A final reflection, prompted by the question "do you think of proportion differently following the last two lessons?" yielded two intriguing findings. Firstly, Luna explained that she now saw "the concept [proportion] in different ways – visual, fractions, ratio... it's just the same thing but in different ways and now I feel able to use it in different ways." The content and delivery of this comment indicates a newly developed self-efficacy which may be interpreted as evidence of a relational schema (Skemp, 1978). Less fortunately, Robert expressed that they felt more confused by proportion and ratio than he had prior to the lessons on ratio. Among the reasons cited by opponents of Skemp's (1978) view is "the great psychological difficulty of restructuring […] existing schemas" (p.13) involved in teaching for relational understanding; while this describes a concern for teachers, it may be the case that this is the obstacle encountered by Robert given his responses during the first interview.

Conclusion

This action research has been undertaken to investigate the potential of a variation theoretic approach to teach Key Stage 3 ratio content. To analyse the viability of this approach to learning, my first research question was “can variation theory support the learning of ratio?” (RQ1). Based on the tacit assumption that proportional reasoning is supported better by relational understanding than instrumental understanding, the second question posed was “in what ways does variation theory support development of a relational understanding of proportion?” (RQ2).

Based on my findings, it is evident that variation theory is a viable approach to the teaching and learning of ratio. The three critical aspects of the object of learning delivered in the two action research lessons – what ratio is, the scaling property of ratio and the applicability of ratio to multiplicative problems – were made accessible and discerned to some extent by all pupils.

The finding that different learners will access critical aspects in different ways – resulting in different types and depths of understanding - supports the literature (e.g. Kullberg et al., 2017). However, there are other takeaways. Firstly, the procedural variation in exercises acted as differentiation; prompting the class to consider the exercises and the presence of patterns afforded certain pupils the opportunity to go beyond the procedural requirements of the exercise and to engage in pattern-spotting and conjecturing. Given the benefits for lower attaining students, it would be interesting to see more definitive research on the relationship between variation theory, cognitive load and working memory.

Turning our attention to the development of relational understanding (RQ2), I conclude that the use of variation theory does indeed support the development of relational understanding. However, it cannot be concluded from this research that variation theory facilitates this more successfully than other pedagogical approaches. It is likely that lower-attaining students benefited more in this sense than higher-attaining students as there was an increase in the number of lower-attaining students attempting questions requiring relational understanding than in previous lessons. Moreover, I conclude that the use of variation theory may have had the impact of, albeit temporarily due to the short time-scale, increasing self-confidence in these pupils. Considering this finding, the impact of variation theory on the affective domain of lower-attaining pupils would be a worthwhile avenue for future research.

References

Allen, R. E. S., & Wiles, J. L. (2016). A rose by any other name: Participants choosing research pseudonyms. *Qualitative Research in Psychology*, *13*(2), 149–165. https://doi.org/10.1080/14780887.2015.1133746

Andrews, P. (1998). Peddling the Myth. *Mathematics in School*, *27*(2), 1–4.

Barton, C. (n.d.). *Activity type 1: Practice*. Retrieved from https://variationtheory.com/practice/

BERA. (2018). *Ethical Guidelines for Educational Research*. BERA London.

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-Media and the Reorganization of Mathematical Thinking: Information and Communication Technologies, Modeling, Visualization and Experimentation*. Springer US. https://doi.org/10.1007/b105001

Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education* (Eighth edition). Routledge.

Collier-Reed, B. I., Ingerman, Å., & Berglund, A. (2009). Reflections on trustworthiness in phenomenographic research: Recognising purpose, context and change in the process of research. *Education as Change*, *13*(2), 339–355. https://doi.org/10.1080/16823200903234901

Fernandez, C., Llinares, S., Modestou, M., & Gagatsis, A. (2010). Proportional reasoning: How task variables influence the development of students' strategies from primary to secondary school. *Acta Didactica Universitatis Comenianae Mathematics*, *10*, 1–18.

Gu, F., Huang, R., & Gu, L. (2017). Theory and Development of Teaching Through Variation in Mathematics in China. In R. Huang & Y. Li (Eds.), *Teaching and Learning Mathematics through Variation* (pp.13–41). SensePublishers. https://doi.org/10.1007/978-94-6300-782-5_2

Jaworski, B. (1999). Tensions in Teachers' Conceptualizations of Mathematics and of Teaching. In L. Burton (Ed.), *Learning and doing mathematics: From hierarchies to networks* (1st ed.). Routledge.

Jerrim, J., & Vignoles, A. (2016). The link between East Asian 'mastery' teaching methods and English children's mathematics skills. *Economics of Education Review*, *50*, 29–44. https://doi.org/10.1016/j.econedurev.2015.11.003

Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM*, *49*(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4

Kvale, S. (1995). The Social Construction of Validity. *Qualitative Inquiry*, *1*(1), 19–40. https://doi.org/10.1177/107780049500100103

Kvale, S. (1996). *Interview Views: An Introduction to Qualitative Research Interviewing*. Sage Publications.

Lai, M. Y., & Murray, S. (2012). Teaching with Procedural Variation: A Chinese Way of Promoting Deep Understanding of Mathematics. *International Journal for Mathematics Teaching and Learning*, 1–25.

Lesh, R., Post, T., & Behr, M. J. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (2nd ed.). Erlbaum.

Lesh, R., Post, T. R., & Behr, M. (1988). Proportional reasoning. In M. Behr, & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp.93-118). National Council of Teachers of Mathematics, Lawrence Erlbaum Associates.

Marton, F., Booth, S., & Booth. (1997). *Learning and awareness*. Lawrence Erlbaum Associates.

Marton, F., & Pang, M. F. (2006). On Some Necessary Conditions of Learning. *Journal of the Learning Sciences*, *15*(2), 193–220. https://doi.org/10.1207/s15327809jls1502_2

Morrison, K. (2013). Interviewing children in uncomfortable settings: 10 lessons for effective practice. *Educational Studies*, *39*(3), 320–337. https://doi.org/10.1080/03055698.2012.760443

Patton, M. Q. (1999). Enhancing the quality and credibility of qualitative analysis. *Health Services Research*, *34*(5 Pt 2), 1189–1208.

Robson, C. (2002). *Real world research: A resource for social scientists and practitioner-researchers*. John Wiley & Sons.

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, *35*(1), 69–87. https://doi.org/10.1080/0305764042000332506

Skemp, R. R. (1978). Relational Understanding and Instrumental Understanding. *The Arithmetic Teacher*, *26*(3), 9–15. https://doi.org/10.5951/AT.26.3.0009

Sweller, J. (2016). Working memory, long-term memory, and instructional design. *Journal of Applied Research in Memory and Cognition*, *5*(4), 360–367.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics*, *16*(2), 181–204. https://doi.org/10.1007/BF02400937

Watson, A., & Mason, J. (2006). Seeing an Exercise as a Single Mathematical Object: Using Variation to Structure Sense-Making. *Mathematical Thinking and Learning*, *8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1